The Composite Fermion Hierarchy: Condensed States of Composite Fermion Excitations?

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A composite Fermion hierarchy theory is constructed in a way related to the original Haldane picture by applying the composite Fermion (CF) transformation to quasiparticles of Jain states. It is shown that the Jain theory coincides with the Haldane hierarchy theory for principal CF fillings. Within the Fermi liquid approach for few electron systems on the sphere a simple interpretation of many-quasiparticle spectra is given and provides an explanation of failure of CF hierarchy picture when applied to the hierarchical 4/11 state.

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The Haldane hierarchy [1,2] of fractional quantum Hall states is obtained when quasiparticle excitations of Laughlin [3] $\nu = 1/m$ incompressible states undergo condensation into new incompressible (daughter) states. Jain [4] noted that certain hierarchy fractions appeared to be particularly stable, and he introduced the composite Fermion (CF) picture which predicts incompressible quantum liquid states at fillings $\nu = n(1+2pn)^{-1}$, where $p = 1, 2, 3, ...,$ and $n = \pm1, \pm2, \pm3, ...$. By invoking electron-hole symmetry condensed CF states at filling factor $1 - |\nu|$ can be found. These two Jain sequences do not contain all odd denominator fractions which are present in the hierarchy theory. The missing odd denominator fractions can be generated by introducing an operator L whose application to a Jain trial wavefunction increases the CF filling factor by unity [5]. Because the application of L to a FQH state weakens it enormously, Jain and Goldman suggest that to a first approximation L can be completely neglected. Instead of constructing trial wavefunctions, Lopez and Fradkin [6] used the CF transformation to construct a conventional many body description of the interactions, both Coulomb and Chern-Simons gauge interactions, among the fluctuations beyond the mean field. They considered only states belonging to the principal Jain sequence, so it is unclear how the hierarchy of odd denominator fractions appears in their approach.

The object of the present paper is to demonstrate how the hierarchy of CF states arises. It involves applying the CF transformation to quasiparticles (CF's in a partially filled shell) and assuming that mean field theory adequately describes the resulting state. The procedure is not at all obvious, since it essentially divides the electrons into different classes. Furthermore, the validity of mean field theory rests on the assumption that the gap for creating new quasiparticles is large compared to their interaction. Direct comparison of the CF hierarchy predictions with numerical results for small systems displays

a number of cases in which this assumption fails.

The state with fractional filling ν_0 has an average of ν_0^{-1} flux quanta per electron. Following Jain, we attach to each electron an even number, $2p_0$, of flux quanta oriented opposite to the applied magnetic field. This gives an effective CF filling factor ν_0^* given by $(\nu_0^*)^{-1} = \nu_0^{-1} - 2p_0$. If ν_0^* is equal to an integer n_1, an integral number of CF levels (or shells) is filled, and the Jain sequence $\nu_0 = n_1(1 + 2p_0 n_1)^{-1}$ is obtained. Negative ν_0^* indicates that the effective magnetic field B^* seen by a CF is oriented opposite to the applied field B.

If ν_0^* is not an integer it can be written as $\nu_0^* = n_1 + \nu_1$, where ν_1 represents the fractional filling of the partially filled CF shell. We define particles in the partially filled CF shell as quasiparticles of the incompressible CF state with filling $\nu_0^* = n_1$ [7]. If excitations between Landau levels are negligible (and closed shells give nothing but constant shift in energy) the lowest energy sector is determined by the energy states of the partially filled shell [8,9]. The same prediction of the incompressible states as for the analogous partially filled electron shell are found when the CF transformation is applied to quasiparticles and $(\nu_1^*)^{-1} = \nu_1^{-1} - 2p_1$. If ν_1^* is still not an integer we can repeat the process which leads us to the sequence of the following equations:

$$\nu_l^{-1} = 2p_l + (n_{l+1} + \nu_{l+1})^{-1}. \tag{1}$$

From Eq. (1), the original fractional electron filling ν_0, can be represented in a form of a continued fraction. If $n_l = 1$ for every value of l, this fraction can be put in the form equivalent to Haldane's [1] (the particle-hole symmetry can be additionally included at every step). The hierarchical incompressible states are predicted when $\nu_{l+1} = 0$ at any step. If $\nu_1 = 0$, the principal Jain sequence is obtained.

It is interesting to see how the Jain principal states arise in the Haldane picture (Eq. (1) with $n_{l+1} = 1$). Let us write the Haldane sequence in the following way

$$\nu_l^{-1} = 2p_l + 1 - (1 + \nu_{l+1}^{-1})^{-1}. \tag{2}$$

Hence, even if all $p_l = 0$, the hierarchical fraction can be generated. After repeating procedure (2) with all $p_l = 0$ (except p_0) till $\nu_{n+1} = 0$ (then $\nu_n = 1$, $\nu_{n-1} = 2$, ..., and $\nu_1 = n$) we have for ν_0:

$$\frac{1}{2p_0+1-\frac{1}{2-\frac{1}{2-\frac{1}{2-\ldots-\frac{1}{2}}}}} = \frac{n}{2p_0 n+1}. \tag{3}$$

Hence, the Jain fraction can be obtained within the sequence (2) and the Haldane hierarchy theory coincides with Jain's for the fraction (3) (the equivalence of different hierarchy schemes was considered in [10–12]). Setting $n_l = 1$ in the sequence (1) does not omit any fraction;

simply the higher values of n are obtained in the more complicated way.

Let us consider a spherical system containing n_l^{QE} quasielectrons at the l^{th} hierarchical level. If those particles occupy one filled Landau shell and there are extra particles in the partially filled second Landau level one finds for the lowest angular momentum shell $2S_{l+1}-1 = n_l^{QE} - n_{l+1}^{QE}$ [1,13]. A condensed state of these quasiparticles is obtained when they fill an integral number of shells at the next step of the hierarchy ($n_{l+2}^{QE}=0$); then we have additionally $2S_{l+1} = (2p_{l+1}+1)(n_{l+1}^{QE}-1)$. Combining these two relations we get:

$$n_l^{QE} = (2p_{l+1}+2)(n_{l+1}^{QE}-1) \tag{4}$$

in agreement with Haldane [1] (if quasiholes are produced then: $n_l^{QE} = 2p_{l+1}(n_{l+1}^{QH}-1)$). Thus, we identify Haldane even numbers as $2p_{l+1}+2$ for quasielectrons ($2p_{l+1}$ for quasiholes), the odd number is given by $2p_0+1$. The construction of the Haldane hierarchy for quasielectrons without applying the CF transformation to them, i. e. $p_{l+1}=0$ (redefinition of the number of quasielectrons), leads to the Jain principal fraction (3), and the picture becomes identical to the original Jain construction. We would like to note that our result is not based on identifying the quasiparticles as Bosons, rather our (equivalent to Haldane results) were obtained treating quasiparticles as Fermions.

Let's look at a simple example for $N=8$ electrons on a sphere. If the degeneracy of the lowest electron shell is $2S_0+1=19$, then we can attach two flux quanta (oriented opposite to B) to each electron to obtain $2S_0^* = 2S_0 - 2(N-1) = 4$. For a finite system we substract $2p_0(N-1)$ flux quanta to obtain the "mean" value of the flux seen be one CF. The lowest CF shell accomodate $2S_0^*+1=5$ particles, the remaining three become quasielectrons in the next shell. These CF excitations have angular momentum $l_{QE}=3$. The allowed values of the total angular momentum $L = 0 \oplus 2 \oplus 3 \oplus 4 \oplus 6$ are obtained by addition of the angular momenta of three Fermions in the $l_{QE}=3$ shell. In numerical calculations these 3QE states are seen as the lowest energy sector of the eight electron spectrum for $2S_0+1=19$ (cf. Fig. 1A). The energy states can be obtained within the Fermi liquid shell model. We can ignore particles in the filled shell and treat only quasiparticles introducing two-body Fermi liquid quasiparticle interaction [7,9]. When we attach 2 flux quanta to each quasiparticle we find new CFs with a new value $2S_{QE}^* = 2S_{QE} - 2(N_{QE}-1) = 2$. Thus the effective shell for these quasielectrons can accomodate $2S_{QE}^*+1=3$ particles, exactly the number we have. Within the CF hierarchy, the filling fraction is determined by $2p_0=2$, $n_1=1$, and ν_1 (for quasielectrons) is given by $2p_1=2$, $n_2=1$, and $\nu_2=0$. From Eq. (1) we find $\nu_1^{-1}=3$ and $\nu_0^{-1}=11/4$.

This picture can be applied to any system consisting of

a finite number of electrons on a spherical surface with a magnetic monopole of strenght $2S$ at its center. For example, for the $N = 6$ and $N = 8$ electron systems the hierarchy of condensed states for values of ν falling between 1 and 2/7 is presented in Table I and Table II. The states in these tables that are marked with stars are not states in the principal Jain sequence. They are hierarchical states which can be obtained by following Haldane's original suggestion and treating the quasiparticle excitations in the same way as the original electrons in the partially filled electron shell were treated. In the Jain-Goldman [5] terminology, these states are obtained with application of the operator L. Of these states only 6/17 and 6/19 states for $N = 6$ and $\nu = 11/17$ and $\nu = 9/31$ states for $N = 8$ have numerically observed the $L = 0$ ground state (but the gaps are very small). It is worth noting that that at fillings of the form $\nu_0 = n_0 + \nu'$, where n_0 is an integer and ν' is an incompressible fractional filling, incompressible states should occur.

The main advantage of the composite fermion theory is that it not only predicts the ground states for the specific values of the magnetic field (S) but it also predicts the lowest energy states for every value of S [9]. Furthermore, when the CF transformation and the MF approximation are applied to N Fermions of angular momentum S, the resulting energy spectra are the same whether the Fermions are electrons, CF quasielectrons or CF quasiholes. Figure 1 gives the energy spectra of three quasielectrons in the angular momentum shells of $S_{QP} = 3.0;\ 3.5;\ 4.0$ (indicated as A, B, C in the figure). Figure 2 presents the cases of four quasielectrons for $S_{QP} = 3.5;\ 4.0;\ 4.5$. Figures 3 and 4 are analogous results for three and four quasihole states. The crosses in these figures are the results of exact numerical diagonalization within the lowest shell (only the lowest energy sector is plotted), the open circles are Fermi liquid theory results [9]. The CF mean field approximation makes the following predictions for the lowest energy states: For three Fermions of angular momentum $S = 3$, an $L = 0$ ground state; for $S = 3.5$ a single quasiparticle in the $L = 1.5$ state; for $S = 4.0$ two quasiparticles each with $l = 2$ should give degenerate states $L = 1 \oplus 3$. For the four Fermion systems, the $S = 3.5$ gives two quasiparticles each with $l = 1.5$ resulting in degenerate states $L = 0 \oplus 2$; $S = 4$ gives a single quasiparticle with $L = 2$; and $S = 4.5$ gives an $L = 0$ ground state. We observe that while CF predictions hold for quasiholes (with small overlap of the $L = 0$ $2QP$ state with the higher energy states in Fig. 4A) they fail to describe low lying quasielectron states (with one exception of $L = 0$ ground state in Fig. 2C).

A simple interpretation of the many-quasielectron spectra comes from noting that QE interaction $V^S_{QE-QE}(J)$ oscillates with J (pair angular momentum) exactly out of phase with the QH interaction [7] (for the same angular momentum shell S). Thus, to some extent,

the shifts in the QE energies due to $QE - QE$ interactions are opposite to those of QH's. We observe, in fact, that states which are highest energy states for quasielectrons are the lowest energy states for quasiholes. The 1/3 quasihole state, which is the 2/7 Jain state, is clearly observed (Fig. 3A, Fig. 4C). Hence, we can not expect the same for the 1/3 state of quasielectrons (4/11 state) and the results for 3QE (Fig. 1A) and 4QE (Fig. 2C) spectra clearly show that, though there is an $L = 0$ ground state for $N = 12$. The qualitative similarity of 3QE and 4QE spectra (Fig. 1 and Fig. 2) to reversed 3QH and 4QH spectra (Fig. 3 and Fig. 4) for other values of $S_{QE} = S_{QH}$ confirms our conclusion.

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TABLE I. $N = 6$

$2S+1$	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
ν	1	–	6/7	–	2/3	–	2/5	–	6/17*	–	1/3	–	6/19*	–	2/7

TABLE II. $N = 8$

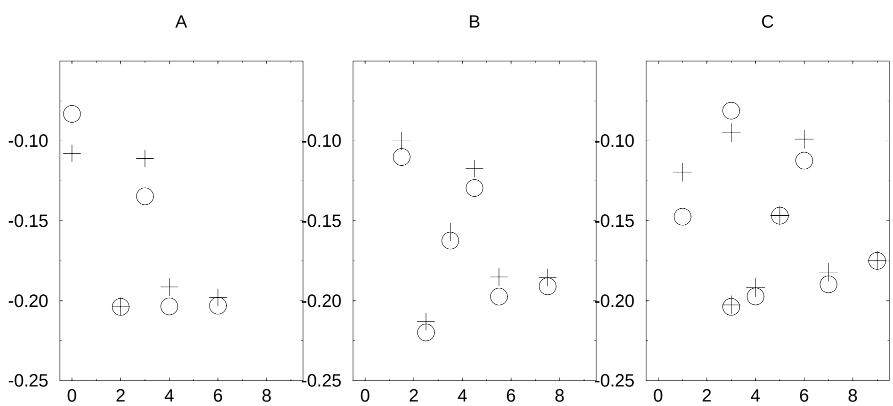
$2S+1$	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27
ν	1	–	8/9	4/5	9/13*	2/3	11/17*	–	11/27*	2/5	9/23*	4/11*	8/23*	–	1/3	–	8/25*	4/13*	9/31*	2/7

FIG. 1. The spectra of three quasielectron systems for $N = 8,\ 9,\ 10$. The quasielectron interaction energy is given in units of e^2/R, where R is the radius of the sphere. The $N = 8$ case (Fig. 1A) is predicted to be the 4/11 state in the hierarchy theory. Circles represent Fermi liquid results and crosses are exact numerical energies.

FIG. 2. The Fermi liquid spectra of four quasielectron systems for $N = 10,\ 11,\ 12$. The $N = 12$ case (Fig. 2C) corresponds to the 4/11 hierarchical state. The results are obtained using interaction parameters found for systems of $N = 8,\ 9,\ 10$, respectively. The agreement with the numerical results is expected to be similar as that for 4QH spectra presented in Fig. 4.

FIG. 3. The spectra of 3QH for $S_{QH} = 3.0;\ 3.5;\ 4.0$ ($N = 4,\ 5,\ 6$). The Fig. 3A ($S_{QH} = 3.0$) shows the 2/7 Jain state.

FIG. 4. The spectra of 4QH for $S_{QH} = 3.5;\ 4.0;\ 4.5$ ($N = 4,\ 5,\ 6$). The interaction parameters were obtained for systems of $N = 6,\ 7,\ 8$, respectively. In Fig. 4C ($S_{QH} = 4.5$) the spectrum of the 2/7 state is presented. The crosses not related to circles come from higher energy states.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B
C
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-0.15
-0.20
-0.25
0
2
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8
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-0.25
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Fig. 1

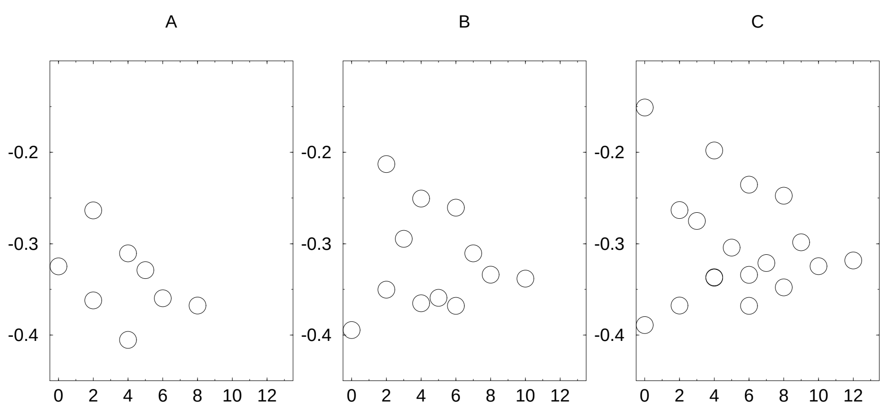

Fig. 2

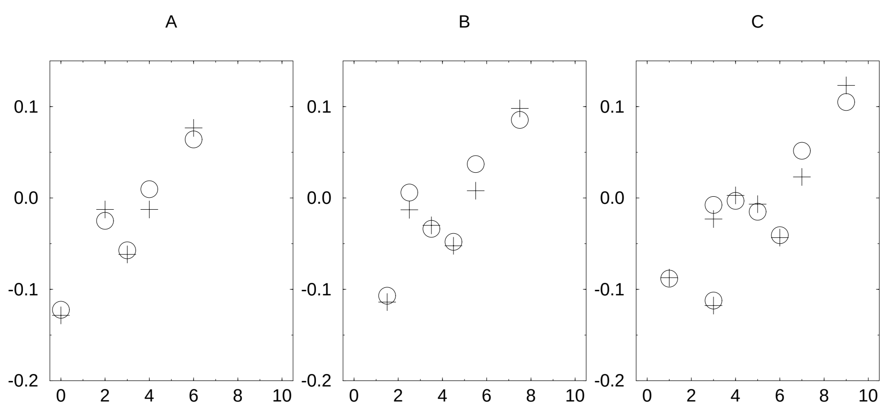

Fig. 3

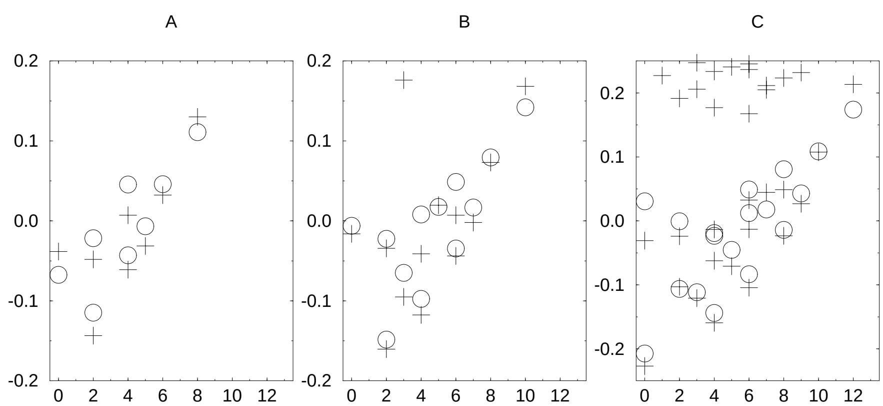

Fig. 4